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S-duality improved perturbation theory in compactified type I/heterotic string theory

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ABSTRACT: We study the mass of the stable non-BPS state in type I/heterotic string theory compactified on a circle with the help of the interpolation formula between weak and strong coupling results. Comparison between the results at different orders indicate that this procedure can determine the mass of the particle to within 10% accuracy over the entire two dimensional moduli space parametrized by the string coupling and the radius of compactification. This allows us to estimate the region of the stability of the particle in this two dimensional moduli space. Outside this region the particle is unstable against decay into three BPS states carrying the same total charge as the original state. We discuss generalization of this analysis to compactification on higher dimensional tori.

KEYWORDS: Superstrings and Heterotic Strings, String Duality

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1 Introduction

Our current understanding of string theory is based mostly on perturbation expansion in the string coupling [1]. Furthermore this perturbation expansion is believed to be an asymptotic expansion. For this reason one might worry that our ability to compute anything in string theory may be limited to very narrow corners of the full string theory landscape — regions in which the theory admits a description as a very weakly coupled string/M/F-theory.

Ref. [2] suggested making use of duality and suitable interpolation formula to translate the weak coupling results in string theory to approximate results for physical quantities over the entire range of string coupling constant. As a specific example, the mass of the stable non-BPS particle in ten dimensional type I/SO(32) heterotic string theory was considered. Using a suitable formula that interpolates between the result for this mass in weakly coupled SO(32) heterotic string theory and weakly coupled type I string theory, an approximate formula for the mass of this state was derived over the entire range of string coupling. Furthermore by comparing the results in different orders it was estimated that this approximate formula lies within 10% of the exact result over the entire range of coupling. Generalization of this analysis has since been discussed in [3–5].

Since a generic string theory moduli space is multi-dimensional it is natural to ask if this interpolation technique can be used to find approximate expressions for various physical quantities over the full multi-dimensional moduli space.¹ In this paper we explore

¹This issue arose already in the analysis of [3] (*albeit* in the context of a supersymmetric gauge theory instead of string theory) where the interpolation technique together with perturbative results were used to determine approximate formulæ for anomalous dimension of non-BPS operators in the complex coupling constant plane.

this in the context of $SO(32)$ heterotic/type I string theory compactified on a circle. If we do not switch on any Wilson line so that the $SO(32)$ gauge group is unbroken then the moduli space is two dimensional, parametrized by the string coupling and the radius of compactification. We use perturbative results in the compactified string theory and a suitable generalization of the interpolation formula used in [2] to derive expression for the mass of the non-BPS state in the full two dimensional moduli space. Comparison between different orders of approximation again indicates that the approximate formula derived here lies within 10% of the exact formula over the entire two dimensional moduli space.

In type I/ $SO(32)$ heterotic string theory compactified on S^1 , one can identify a set of BPS states whose total charge is the same as that of the charge carried by the non-BPS state under study. Thus the latter can decay into the former if the mass of the non-BPS state is larger than the sum of the masses of the BPS states to which it could possibly decay. With the help of the approximate formula for the mass we determine the part of the region of the two dimensional moduli space in which the non-BPS state is unstable. Again we find that the region determined this way is only mildly sensitive to the order of the approximation that we use.

The rest of the paper is organized as follows. In section 2 we review the interpolation technique and the normalization conventions of [2], and the additional ingredients we need for dealing with the compactified theory. In section 3 we carry out the computation of the one loop correction to the mass of stable non-BPS state in type I string theory compactified on S^1 . In section 4 we carry out a similar calculation in $SO(32)$ heterotic string theory compactified on S^1 . In section 5 we construct the interpolation formula for the mass of the non-BPS state in various approximation, and compare the results in different orders of approximation. In section 6 we use the interpolation formula to analyze the region of stability of the non-BPS state. In section 7 we discuss extension of our analysis to the case of compactification on higher dimensional tori. We conclude in section 8 with a brief summary of our results and their possible relation to related developments.

2 Normalization conventions and tree level results

We shall use the normalization conventions used in [2]. The purpose of this section will be to review these conventions and also introduce the extra conventions involving the radius of compactification.

Let g_H and g_I be the string coupling in ten dimensional heterotic and type I string theories respectively. We introduce a new coupling parameter g in terms of which g_H and g_I are given by

$$g_H = 2^{7/2} \pi^{7/2} g, \quad g_I = 2^{3/2} \pi^{7/2} g^{-1}. \quad (2.1)$$

We normalize the heterotic and type I metric such that the heterotic string tension in heterotic metric and the type I string tension in type I metric are both given by $1/2\pi$. The mass of the non-BPS state, measured in the ten dimensional Einstein metric, is parametrized by

$$M(g) = 2^{15/8} \pi^{7/8} F(g). \quad (2.2)$$

The tree level weak and strong coupling values of $F(g)$, computed respectively from tree level heterotic and type I string theories, are

$$F_0^W(g) = g^{1/4}, \quad F_0^S(g) = g^{3/4}. \quad (2.3)$$

Upon compactification on a circle the tree level masses will continue to be given by (2.3) if we measure it in the canonical metric in ten dimensions. We shall follow this convention. Let r_I , r_H and r_E denote the radii of the compact circle measured in the type I, heterotic and canonical metric. Then

$$\begin{aligned} r_E &= (g_I)^{-1/4} r_I = 2^{-3/8} \pi^{-7/8} g^{1/4} r_I, \\ r_H &= (g_H)^{1/4} r_E = 2^{7/8} \pi^{7/8} g^{1/4} r_E = 2^{1/2} g^{1/2} r_I. \end{aligned} \quad (2.4)$$

We expect the quantum corrections in the heterotic and type I string theories to modify the weak coupling results to

$$F^W(g) = g^{1/4} \left[1 + \sum_{k=1}^{\infty} A_{2k}(r_H) g^{2k} \right], \quad F^S(g) = g^{3/4} \left[1 + \sum_{n=1}^{\infty} B_n(r_I) g^{-n} \right], \quad (2.5)$$

where the functions $A_{2k}(r_H)$ and $B_n(r_I)$ have finite $r_H \rightarrow \infty$ and $r_I \rightarrow \infty$ limits respectively, corresponding to the results in the non-compact theory. We now introduce the interpolating function²

$$\begin{aligned} F_{m,n}(g) &= g^{1/4} \left[1 + a_1(r_H)g + \cdots a_m(r_H)g^m + b_n(r_I)g^{m+1} + b_{n-1}(r_I)g^{m+2} \right. \\ &\quad \left. + \cdots + b_1(r_I)g^{m+n} + g^{m+n+1} \right]^{1/\{2(m+n+1)\}}, \end{aligned} \quad (2.6)$$

where the functions $a_k(r_H)$ and $b_k(r_I)$ are determined as follows. We determine $a_k(r_H)$ by demanding that after setting the b_k 's to zero, the expansion of (2.6) in powers of g at fixed r_H agrees with that of $F^W(g)$ up to order $g^{\frac{1}{4}+m}$. Similarly the functions $b_k(r_I)$ are determined by demanding that after setting the $a_k(r_H)$'s to zero, the expansion of (2.6) in powers of g^{-1} at fixed r_I agrees with that of $F^S(g)$ up to order $g^{\frac{3}{4}-n}$.

We shall now argue that the weak coupling expansion at fixed r_H of the full function $F_{m,n}$ keeping both a_k 's and b_k 's non-zero coincides with that of F^W up to order $g^{\frac{1}{4}+m}$, and similarly the strong coupling expansion at fixed r_I of the full function $F_{m,n}$ coincides with that of F^S up to order $g^{\frac{3}{4}-n}$. From eq. (2.4) it follows that $r_I = 2^{-1/2} g^{-1/2} r_H$, and hence as $g \rightarrow 0$ keeping r_H fixed, $r_I \rightarrow \infty$. In this limit the coefficients B_k appearing

²There are various other possible interpolation schemes (see e.g. [6, 7]), but the one given in (2.6), called the fractional power of polynomial (FPP) scheme in [3], seems to be most suitable for our purpose as this gives a clear separation between the coefficients which are determined using weak coupling expansion and the coefficients which are determined using strong coupling expansion. This is needed to ensure that the weak coupling expansions *at fixed* r_H and strong coupling expansion *at fixed* r_I match the perturbation expansions. The difficulty in achieving this with other approximation schemes, e.g. 2-point Padé approximant, is similar to the difficulties faced in [3] in getting a duality invariant approximation scheme beyond four loops using 2-point Padé approximant.

in the strong coupling expansion should approach finite values given by the results in the non-compact theory. Thus the coefficients b_k , determined in terms of the coefficients B_ℓ for $k \leq n$, should also approach finite values in this limit. This shows that the expansion of b_k in powers of g at fixed r_H contains non-negative powers of g . Substituting this into (2.6) we now see that the coefficients b_k do not affect the weak coupling expansion of $F_{m,n}$ to order $g^{\frac{1}{4}+m}$, and hence the weak coupling expansion of $F_{m,n}$ to this order agrees with that of F^W . A similar analysis shows that the expansion of a_k in powers of g^{-1} at fixed r_I contains non-positive powers of g . Hence the expansion of $F_{m,n}$ in powers of g^{-1} at fixed r_I to order $g^{\frac{3}{4}-n}$ is insensitive to the coefficients a_k and coincides with that of $F^S(g)$.

From (2.3), (2.6) we can find the following interpolating functions for the mass of the non-BPS particle

$$\begin{aligned} F_{0,0}(g) &= g^{1/4} (1+g)^{1/2}, \\ F_{1,0}(g) &= g^{1/4} (1+g^2)^{1/4}. \end{aligned} \quad (2.7)$$

3 Strong coupling expansion

Denote by ΔM the first order correction to the mass formula from the strong coupling end, i.e. in weakly coupled type I string theory compactified on a circle S^1 of radius r_I . This can be obtained by calculating the one loop correction to the energy of the non-BPS D0-brane of type I string theory compactified on S^1 . This calculation differs from the corresponding calculation in [2] by having to include extra contribution from open string winding modes along the circle, beginning on a D0-brane and ending on one of its images. The result takes the form [8–11]

$$-\Delta M = \frac{1}{2} g_I^{\frac{1}{4}} (8\pi^2)^{-\frac{1}{2}} \int_0^\infty s^{-\frac{3}{2}} ds [Z_{NS;D0D0} - Z_{R;D0D0} + Z_{NS;D0D9} - Z_{R;D0D9}], \quad (3.1)$$

where $Z_{NS;D0D0}$, $Z_{R;D0D0}$, $Z_{NS;D0D9}$, $Z_{R;D0D9}$ denote respectively the contributions from the NS and R sector open strings with both ends on the D0-brane and NS and R open strings with one end on the D0-brane and the other end on the D9-brane wrapped on S^1 of radius r_I . Explicit computation gives

$$\begin{aligned} Z_{NS;D0D0} &= \frac{1}{2} \left(\sum_n \tilde{q}^{2n^2 r_I^2} \right) \frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8} + 2^{\frac{5}{2}} (1-i) \frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{\frac{5}{2}} (1+i) \frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})}, \\ Z_{R;D0D0} &= \frac{1}{2} \left(\sum_n \tilde{q}^{2n^2 r_I^2} \right) \frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8}, \\ Z_{NS;D0D9} &= 16\sqrt{2} \frac{f_2(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_3(\tilde{q})}, \\ Z_{R;D0D9} &= 16\sqrt{2} \frac{f_3(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_2(\tilde{q})}, \end{aligned} \quad (3.2)$$

where n is the quantum number corresponding to the winding number of the fundamental open string along the compact direction and

$$\begin{aligned}
 \tilde{q} &\equiv e^{-\pi s}, \\
 f_1(q) &= q^{1/12} \prod_{n=1}^{\infty} (1 - q^{2n}) = \eta(2\tau), \quad q \equiv e^{2\pi i \tau}, \\
 f_2(q) &= \sqrt{2} q^{1/12} \prod_{n=1}^{\infty} (1 + q^{2n}) = \sqrt{2} \frac{\eta(4\tau)}{\eta(2\tau)}, \\
 f_3(q) &= q^{-1/24} \prod_{n=1}^{\infty} (1 + q^{2n-1}) = \frac{\eta(2\tau)^2}{\eta(\tau)\eta(4\tau)}, \\
 f_4(q) &= q^{-1/24} \prod_{n=1}^{\infty} (1 - q^{2n-1}) = \frac{\eta(\tau)}{\eta(2\tau)}.
 \end{aligned} \tag{3.3}$$

Individual terms in (3.1) are both IR and UV divergent. Using the prescription for the IR and UV regularization described in [2] we can express (3.1) as follows,

$$\Delta M = \tilde{K}_S (g_I)^{1/4}, \tag{3.5}$$

$$\begin{aligned}
 \tilde{K}_S &\equiv -\frac{1}{2} (8\pi^2)^{-\frac{1}{2}} \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left[\int_{\epsilon}^{\Lambda} s^{-\frac{3}{2}} ds \left\{ \frac{1}{2} \left(\sum_n \tilde{q}^{2n^2 r_I^2} \right) \left(\frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8} - \frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8} \right) \right. \right. \\
 &\quad \left. \left. + 16\sqrt{2} \frac{f_2(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_3(\tilde{q})} - 16\sqrt{2} \frac{f_3(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_2(\tilde{q})} \right\} \right. \\
 &\quad \left. + \int_{\epsilon/4}^{\Lambda} s^{-\frac{3}{2}} ds \left\{ 2^{\frac{5}{2}} (1 - i) \frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{\frac{5}{2}} (1 + i) \frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})} \right\} \right].
 \end{aligned} \tag{3.6}$$

Note that for $r_I < 1/\sqrt{2}$, the $n = 1$ term in the sum behaves as $\tilde{q}^{2r_I^2-1} = e^{\pi s(1-2r_I^2)}$ and hence the integral over s has a divergence from the large s region. This reflects the appearance of the open string tachyon in the spectrum for $r_I < 1/\sqrt{2}$ [12]. For this reason the open string loop corrections to the mass of stable non-BPS state makes sense only for $r_I \geq 1/\sqrt{2}$, and in the rest of this section we shall focus on this region. Using (2.4) we see that in terms of the radius r_H in the heterotic metric, this condition takes the form

$$r_H > g^{1/2}. \tag{3.7}$$

It is possible to convert expression (3.6) in the ‘closed string channel’ using the modular tranformation laws of f_i ’s:

$$\tilde{K}_S = - \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \frac{1}{4\pi} (8\pi^2)^{-\frac{1}{2}} \left[\int_{\pi/\Lambda}^{\pi/\epsilon} dt (C_{00} + C_{09} + C_{09}^*) + \int_{\pi/4\Lambda}^{\pi/\epsilon} dt (\mathcal{M} + \mathcal{M}^*) \right] \tag{3.8}$$

where

$$C_{00} = \left(\frac{\pi}{t} \right)^4 \left(\sum_n \frac{q^{\frac{n^2}{2r_I^2}}}{\sqrt{2} r_I} \right) \left(\frac{f_3(q)^8}{f_1(q)^8} - \frac{f_4(q)^8}{f_1(q)^8} \right),$$

$$\begin{aligned}
C_{09} &= 2^{\frac{9}{2}} \left(\frac{f_4(q)^9 f_1(q)}{f_2(q)^9 f_3(q)} - \frac{f_3(q)^9 f_1(q)}{f_2(q)^9 f_4(q)} \right), \\
\mathcal{M} &= 2^{\frac{9}{2}} \left(\frac{f_3(iq)^9 f_1(iq)}{f_2(iq)^9 f_4(iq)} - \frac{f_4(iq)^9 f_1(iq)}{f_2(iq)^9 f_3(iq)} \right), \\
q &\equiv e^{-t}.
\end{aligned} \tag{3.9}$$

C_{00} denotes the cylinder amplitude with boundaries lying on the D0-brane, given by the inner product between the boundary states of D0-brane. C_{09} denotes the cylinder amplitude with one boundary lying on the D0-brane and the other boundary on the D9-brane wrapped on S^1 , given by the inner product between the boundary states of D0-brane and the D9-brane wrapped on S^1 . \mathcal{M} denotes the möbius strip amplitude with boundary lying on the D0-brane, given by the inner product between the boundary states of D0-brane and the crosscap.

Using this and eqs. (2.1), (2.2) we can write the corrected strong coupling expression for $F^S(g, r_I)$ to order $g^{\frac{3}{4}-1}$ as,

$$F_1^S(g, r_I) = g^{\frac{3}{4}} \left(1 + K_S(r_I) g^{-1} \right), \quad K_S(r_I) \equiv 2^{-\frac{3}{2}} \tilde{K}_S. \tag{3.10}$$

$K_S(r_I)$ can be obtained by integrating expression (3.6) numerically for different values of r_I . We find that the result of this numerical evaluation fits well with the function

$$K_S(r_I) \simeq 0.351 - 0.048 \exp \left[-10 (r_I - 2^{-1/2})^{2/3} \right], \tag{3.11}$$

within 1% accuracy over the entire range $1/\sqrt{2} \leq r_I < \infty$.

4 Weak coupling expansion

First order correction to M in the weakly coupled heterotic string theory given by

$$\delta M = M K_W(r_H) g^2, \tag{4.1}$$

where $K_W(r_H)$ can be calculated by doing a one loop heterotic string calculation similar to that in [2], but including the effect of closed heterotic string winding and momentum modes along the circle. The result is

$$\begin{aligned}
K_W(r_H) &= -\frac{1}{64\pi} \int d^2\tau \int d^2z \left[\left\{ \sum_{\nu'} \overline{\left\{ \vartheta_{\nu'} \left(\frac{z}{2} \right) \right\}^{16}} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} \left(\frac{\vartheta_{11}(z)}{\vartheta_{11}(z)} \right)^2 \right\} \right. \\
&\quad \times \exp \left(-\frac{4\pi z_2^2}{\tau_2} \right) (\tau_2)^{-9/2} \frac{1}{r_H} \left\{ \sum_{n,w} \exp \left(-\frac{\pi i \bar{\tau}}{2} \left(\frac{n}{r_H} + w r_H \right)^2 + \frac{\pi i \tau}{2} \left(\frac{n}{r_H} - w r_H \right)^2 \right) \right\} \Bigg],
\end{aligned} \tag{4.2}$$

with τ denoting the modular parameter of the torus, ν denoting the spin structure on the torus taking values 00, 01, 10 and 11, ϑ are the Jacobi theta functions, r_H radius of S^1 on which heterotic string theory is compactified and n, w representing the momentum and winding number along the compactified direction. Since this expression is invariant under

T-duality transformation $r_H \rightarrow 1/r_H$ (except for the overall factor of $1/r_H$ that is taken care of by the transformation law of g^2 multiplying it), we can focus on the region $r_H \geq 1$. In this region the evaluation of the integral can be facilitated using a Poisson resummation in the variable n . This yields

$$K_W(r_H) = -\frac{1}{64\pi} \int d^2\tau \int d^2z \left[\left\{ \sum_{\nu'} \left\{ \overline{\vartheta_{\nu'}} \left(\frac{z}{2} \right)^{16} \right\} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} \left(\frac{\vartheta_{11}(z)}{\vartheta_{11}(\overline{z})} \right)^2 \right\} \right. \\ \left. \times \exp \left(-\frac{4\pi z_2^2}{\tau_2} \right) (\tau_2)^{-5} \left\{ \sum_{k,w} \exp \left(-\frac{\pi}{\tau_2} r_H^2 |k - w\tau|^2 \right) \right\} \right]. \quad (4.3)$$

In the $r_H \rightarrow \infty$ limit only $k = w = 0$ term in the sum survives, giving back the ten dimensional result. For finite r_H numerically integrating expression (4.3) for different values of r_H we find that the result can be fitted approximately with the function,

$$K_W(r_H) \simeq 0.23 \left(1 + \frac{1}{r_H^7} \right)^{2/7}. \quad (4.4)$$

Then the corrected weak coupling expression for $F^W(g, r_H)$ to order g^2 is given as,

$$F_2^W(g, r_H) = g^{\frac{1}{4}} \left(1 + K_W(r_H) g^2 \right). \quad (4.5)$$

Notice that $K_W(r_H)$ diverges in the $r_H \rightarrow 0$ limit. This is easily understood using the known T-duality invariance $r_H \rightarrow 1/r_H$ in the heterotic string theory. Under this the ten dimensional string coupling transforms to g/r_H . Defining

$$\tilde{r}_H = \frac{1}{r_H}, \quad \tilde{g} = \frac{g}{r_H}, \quad (4.6)$$

we can express (4.5) as

$$F_2^W(g, r_H) = (\tilde{r}_H)^{-1/4} \tilde{g}^{\frac{1}{4}} \left(1 + .23 (1 + (\tilde{r}_H)^{-7})^{2/7} \tilde{g}^2 \right). \quad (4.7)$$

Except for the overall factor of $(\tilde{r}_H)^{-1/4}$ which reflects the overall scale factor relating the ten dimensional Einstein metric in the dual pair of heterotic string theories, we see that this has a perfectly good $\tilde{r}_H \rightarrow \infty$ ($r_H \rightarrow 0$) limit at fixed \tilde{g} . For this reason, for $r_H < 1$ it is more natural to use the coupling constant \tilde{g} of the T-dual theory as an expansion parameter.

5 Interpolating functions

We now turn to the construction of the interpolating functions. For definiteness we shall treat g and r_H as independent variables. We shall divide up the g - r_H plane into several regions shown in figure 1 and use different interpolating functions in these different regions.

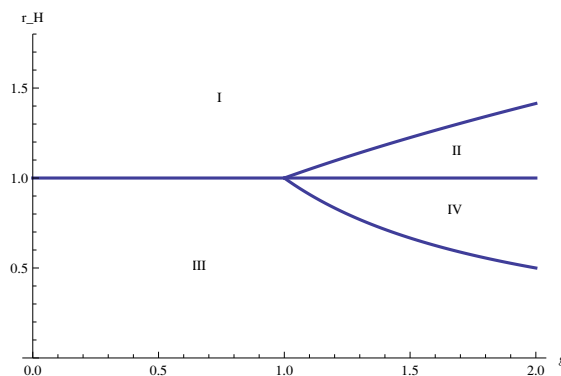


Figure 1. The four regions in the g - r_H plane. The curves bounding these regions are $r_H = 1$, $r_H = g^{1/2}$ and $r_H = g^{-1}$.

Region I: first consider the region I defined by

$$\mathbf{I} : r_H \geq 1, \quad r_H \geq g^{1/2}. \quad (5.1)$$

In this region $r_I \geq 1/\sqrt{2}$ (see (3.7)) and both the heterotic perturbation theory in powers of g and type I perturbation theory in powers of g^{-1} are well defined at small and large g respectively. Thus we can use standard interpolation formula described in section 2:

$$F_{0,0}(g, r_H) = g^{1/4} (1 + g)^{1/2}, \quad (5.2)$$

$$F_{1,0}(g, r_H) = g^{1/4} (1 + g^2)^{1/4}, \quad (5.3)$$

$$F_{0,1}(g, r_H) = g^{1/4} (1 + 4 K_S(r_I)g + g^2)^{1/4}, \quad (5.4)$$

$$F_{1,1}(g, r_H) = g^{1/4} (1 + 6 K_S(r_I)g^2 + g^3)^{1/6}, \quad (5.5)$$

$$F_{2,0}(g, r_H) = g^{1/4} (1 + 6 K_W(r_H)g^2 + g^3)^{1/6}, \quad (5.6)$$

$$F_{2,1}(g, r_H) = g^{1/4} (1 + 8 K_W(r_H)g^2 + 8 K_S(r_I)g^3 + g^4)^{1/8}, \quad (5.7)$$

$$F_{3,0}(g, r_H) = g^{1/4} (1 + 8 K_W(r_H)g^2 + g^4)^{1/8}, \quad (5.8)$$

and

$$F_{3,1}(g, r_H) = g^{1/4} (1 + 10 K_W(r_H)g^2 + 10 K_S(r_I)g^4 + g^5)^{1/10}. \quad (5.9)$$

Region II: region II is defined by

$$\mathbf{II} : 1 \leq r_H \leq g^{1/2}. \quad (5.10)$$

In this region heterotic perturbation theory in power of g is still valid for small g but type I perturbation theory in powers of g^{-1} breaks down at large g due to the presence of the tachyon. Thus the only interpolating functions we can use are $F_{m,0}$ for $0 \leq m \leq 3$.

Region III: region III is defined by

$$\mathbf{III} : r_H \leq 1, \quad r_H \leq g^{-1}. \quad (5.11)$$

The significance of this region can be understood by reexpressing (5.11) in terms of T-dual variables \tilde{r}_H, \tilde{g} introduced in (4.6). This corresponds to

$$\tilde{r}_H \geq 1, \quad \tilde{r}_H \geq \tilde{g}^{1/2}. \quad (5.12)$$

In this region we shall use the interpolating functions of region I with (g, r_H) replaced by (\tilde{g}, \tilde{r}_H) and with an overall multiplicative factor of $(\tilde{r}_H)^{-1/4} = (r_H)^{1/4}$ to account for the rescaling of the canonical metric discussed below (4.7). Thus we use the functions

$$\tilde{F}_{m,n}(g, r_H) = (r_H)^{1/4} F_{m,n}(g/r_H, 1/r_H). \quad (5.13)$$

Physically this corresponds to using an interpolating formula between the T-dual heterotic string theory and its strong coupling dual type I string theory (obtained in the $\tilde{g} \rightarrow \infty$ limit at fixed $\tilde{r}_H/\tilde{g}^{1/2}$). This is not the original type I string theory, but related to it via a strong-weak coupling duality transformation. This is apparent from the fact that while in the original type I string theory the non-BPS D-brane develops a tachyon for $r_H < g^{1/2}$, in the new theory the non-BPS D-brane is tachyon free for $r_H < g^{-1}$.

Region IV: region IV is defined by

$$\text{IV} : g^{-1} \leq r_H \leq 1. \quad (5.14)$$

In the (\tilde{g}, \tilde{r}_H) variables this corresponds to $1 \leq \tilde{r}_H \leq \tilde{g}^{-1/2}$, i.e. this is the heterotic T-dual image of region II. Thus we use the interpolation formulæ of region II with (g, r_H) replaced by (\tilde{g}, \tilde{r}_H) :

$$\tilde{F}_{m,0}(g, r_H) = (r_H)^{1/4} F_{m,0}(g/r_H, 1/r_H). \quad (5.15)$$

Note that the results in regions III and IV can be obtained from those in regions I and II by heterotic T-duality transformation (4.6). For this reason we shall focus on regions I and II from now on.

In figure 2 we have plotted the ratios of $F_{m,n}$ to $F_{3,1}$ as a function of g for four different values of r_H in region I. As we can see, except for $F_{0,0}$, all other $F_{m,n}$'s remain within about 10% of $F_{3,1}$ over the entire allowed range of g in region I. This suggests that $F_{3,1}$ gives the actual mass of the particle within about 10% error over the entire range of parameter space of region I (and hence also of region III). We shall return to a discussion of regions II and IV in the next section.

6 Stability analysis

Let (n, w) denote the momentum and winding numbers of a heterotic string state along the compact circle. The non-BPS state carries the same quantum numbers as that of an heterotic string state with quantum numbers $(n, w) = (1, 1)$ in the spinor representation of $\text{SO}(32)$, a state with quantum number $(n, w) = (0, -1)$ in the adjoint/singlet representation of $\text{SO}(32)$ and a state with quantum number $(n, w) = (-1, 0)$ in the singlet/adjoint

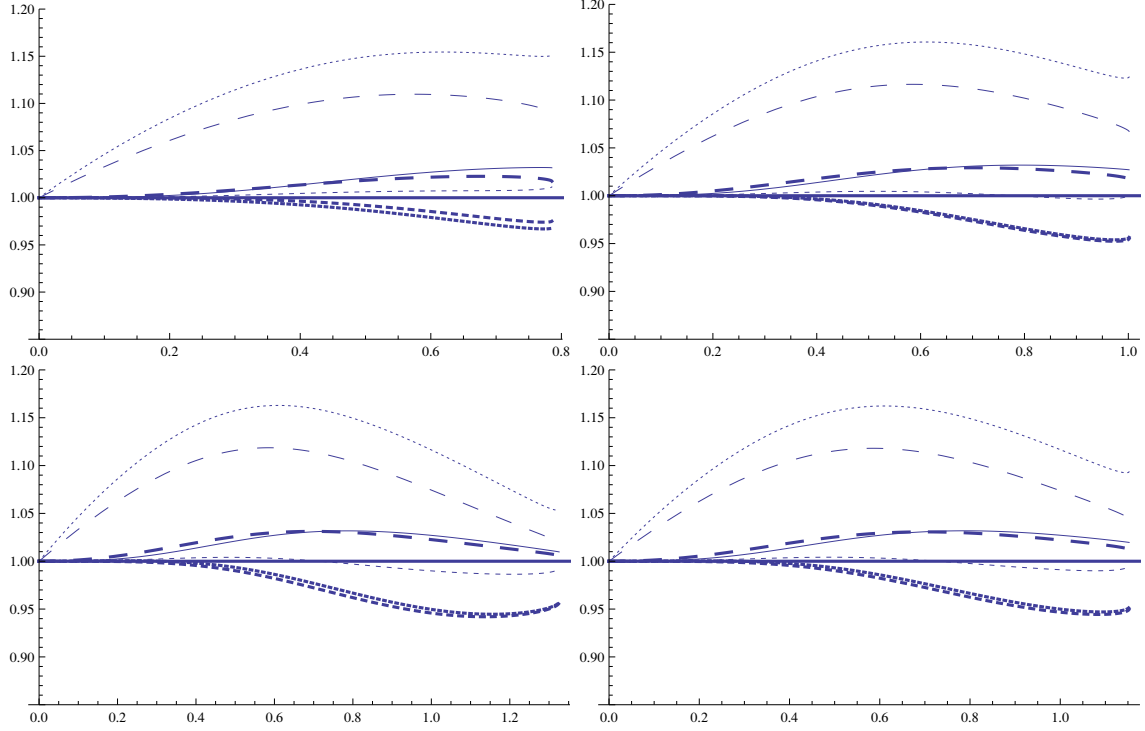


Figure 2. Graph of $F_{m,n}(g)/F_{3,1}(g)$ vs. $\tan^{-1} g$ for various (m,n) in region I. The labels are as follows: thin dots for $F_{0,0}$, thick dots for $F_{1,0}$, small thin dashes for $F_{2,0}$, small thick dashes for $F_{3,0}$, large thin dashes for $F_{0,1}$, large thick dashes for $F_{1,1}$, continuous thin line for $F_{2,1}$ and continuous thick line for $F_{3,1}$. The four graphs, clockwise from top left, correspond to $r_H = 1, 1.25, 1.5$ and 2 respectively.

representation.³ The total mass of this state in the heterotic string metric for $r_H > 1$ is

$$M_{BPS}^H = \left(r_H + \frac{1}{r_H} \right) + r_H + \frac{1}{r_H} = 2 \left(r_H + \frac{1}{r_H} \right). \quad (6.1)$$

In the Einstein metric this is given by

$$M_{BPS}^E = (g_H)^{1/4} M_{BPS}^H = 2^{15/8} \pi^{7/8} g^{1/4} \left(r_H + \frac{1}{r_H} \right). \quad (6.2)$$

After taking into account the normalization (2.2) we get

$$F_{BPS}(g) \equiv 2^{-15/8} \pi^{-7/8} M_{BPS}^E = g^{1/4} \left(r_H + \frac{1}{r_H} \right) = 2^{1/2} g^{3/4} \left(r_I + \frac{1}{2g r_I} \right). \quad (6.3)$$

This expression is not renormalized. Furthermore it is manifestly invariant under heterotic T-duality transformation (4.6).

³We could also consider decay into $(n, w) = (1, 1)$ in the spinor representation and $(n, w) = (-1, -1)$ in the adjoint or singlet representation, or $(n, w) = (1, 1)$ in the spinor representation, $(n, w) = (1, -1)$ in the singlet representation and $(n, w) = (-2, 0)$ in the singlet or adjoint representation. In each of these cases the total mass of the decay products is the same as that given on the right hand side of (6.1).

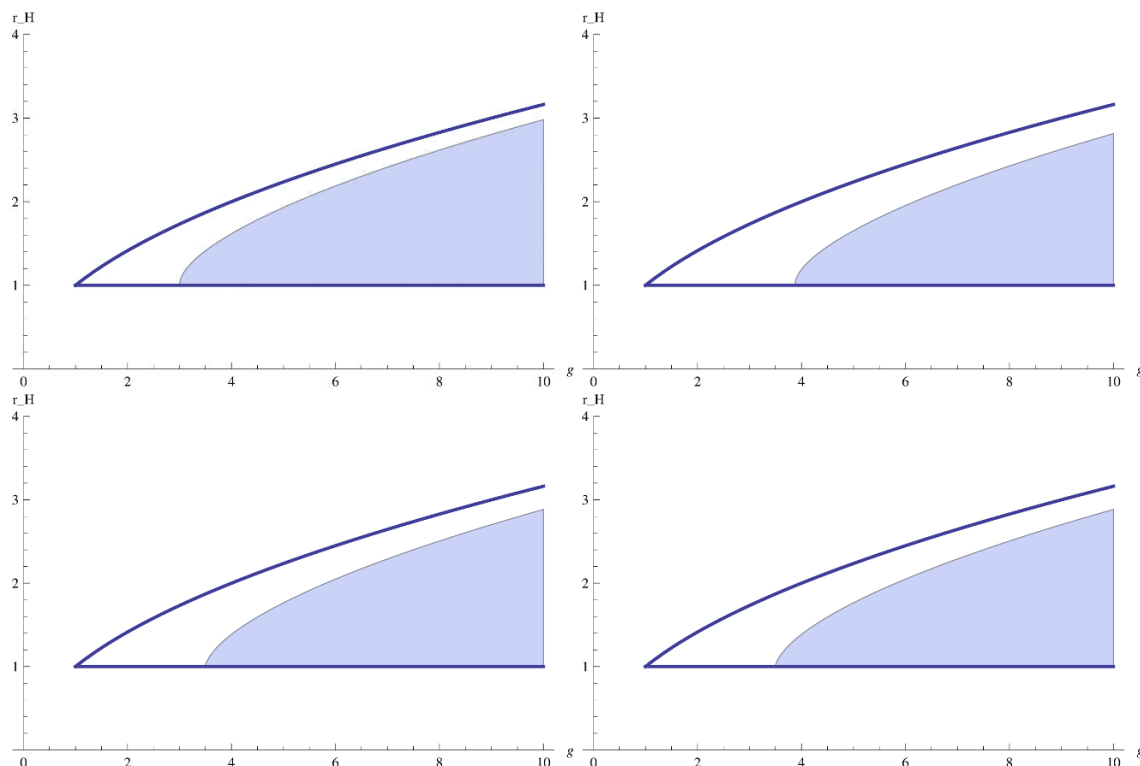


Figure 3. Region of instability of $F_{m,0}$ for $0 \leq m \leq 3$ in the region II of the $g - r_H$ plane. Clockwise from top left the diagrams are based on the interpolating function $F_{0,0}$, $F_{1,0}$, $F_{2,0}$ and $F_{3,0}$ respectively.

Perturbative stability of non-BPS states in compactified SO(32) heterotic string theory has been analyzed in [13]. We shall now analyze non-perturbative stability of these states using the interpolation formula. In the (m, n) approximation the non-BPS particle is stable when its mass is less than the total mass of the BPS constituents to which it can decay. In regions I and II this requires

$$F_{m,n}(g) < g^{1/4} \left(r_H + \frac{1}{r_H} \right). \quad (6.4)$$

In regions III and IV the left hand side is replaced by $\tilde{F}_{m,n}(g)$, but the results in these regions are related to those in regions I and II respectively by heterotic T-duality. Now one can check explicitly that in region I all the $F_{m,n}(g)$'s satisfy (6.4), showing that whatever approximation we use, the non-BPS state is stable in this region. By heterotic T-duality the same result holds in region III. In region II only $F_{m,0}$ approximations make sense. In figure 3 we have shown by the shaded region the region of instability of the non-BPS state in different approximations. As we can see, these regions are not too different from each other, indicating that this is a fairly good approximation to the true region of instability of the non-BPS state in region II. The region of instability in region III can be found from this using heterotic T-duality transformation.

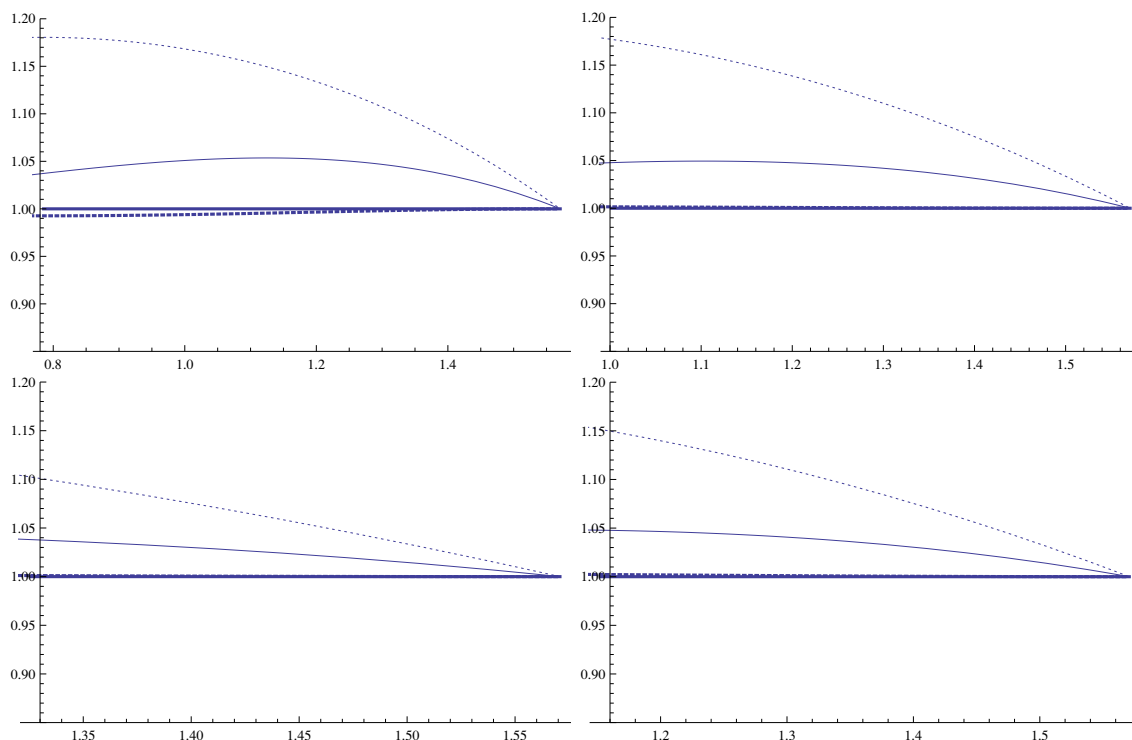


Figure 4. Graph of $F_{m,0}(g)/F_{3,0}(g)$ vs. $\tan^{-1} g$ for various m in region II. The labels are as follows: thin dots for $F_{0,0}$, thick dots for $F_{1,0}$, continuous thin line for $F_{2,0}$ and continuous thick line for $F_{3,0}$. The four graphs, clockwise from top left, correspond to $r_H = 1, 1.25, 1.5$ and 2 respectively.

One point worth noticing is that in each of these plots, there is a narrow strip of region II where the non-BPS state is stable. This may seem a bit surprising at first since in the whole of region II perturbative open string theory describing the non-BPS D0-brane develops a tachyon. Note however that this is true only in tree level open string theory which corresponds to $g \rightarrow \infty$ limit of this diagram. Indeed in this limit the strip width reduces to zero showing that the D0-brane becomes unstable as soon as we cross the upper boundary of region II. At finite g however the tachyon mass² itself may get corrected and hence the tachyon may not develop as soon as we cross the upper boundary. We cannot do this analysis directly in type I string theory since at present it is not understood how to carry out open string perturbation theory in the presence of a tree level tachyon. Instead we have checked the stability by comparing the mass of the unstable brane with the total mass of the decay products, and arrived at figure 3.

It is natural to ask whether one can reliably determine the mass of the non-BPS particle in the white part of region II where it is expected to be stable. This can be done via the interpolating function $F_{m,0}$. Figure 4 shows the ratios of $F_{m,0}$ to $F_{3,0}$ in region II. As we can see from this graph, the ratios remain within about 10% of unity except for $F_{0,0}$, indicating that the interpolating formulæ based on $F_{3,0}$ determines the actual mass of the stable non-BPS particle even inside region II to within about 10% accuracy.

7 Compactification on higher dimensional tori

In this section we shall briefly discuss the generalization of the above analysis to type I/SO(32) heterotic string theory compactified on T^d — a d dimensional torus. We shall refrain from switching on any gauge field background so as to keep SO(32) gauge group unbroken, but allow generic values of the other moduli. This corresponds to choosing arbitrary constant metric G_{Hmn} and NS-NS 2-form field B_{Hmn} along T^d in heterotic description, and arbitrary constant metric G_{Imn} and RR 2-form field C_{Imn} in the type I description. Generalization of (2.4) relating the two sets of moduli are

$$G_{Hmn} = 2g G_{Imn}, \quad B_{Hmn} = C_{Imn}. \quad (7.1)$$

The weak and strong coupling expansions take the same form as in (2.5) and the interpolation formula takes the form of (2.6) with the dependence on r_H now generalized to dependence on G_{Hmn} and B_{Hmn} and the dependence on r_I generalized to dependence on G_{Imn} . The procedure for constructing the coefficients a_i and b_i in (2.6) is the same as that for S^1 compactification.

The analog of the strong coupling expansion (3.6), (3.10) now takes the form:

$$\begin{aligned} F_1^S(g) &= g^{\frac{3}{4}} \left(1 + K_S g^{-1} \right), \\ K_S &\equiv -2^{-5/2} (8\pi^2)^{-\frac{1}{2}} \lim_{\Lambda \rightarrow \infty} \lim_{\epsilon \rightarrow 0} \left[\int_{\epsilon}^{\Lambda} s^{-\frac{3}{2}} ds \left\{ \frac{1}{2} \left(\sum_{\vec{n}} \tilde{q}^{2G_{Ikl} n^k n^l} \right) \left(\frac{f_3(\tilde{q})^8}{f_1(\tilde{q})^8} - \frac{f_2(\tilde{q})^8}{f_1(\tilde{q})^8} \right) \right. \right. \\ &\quad \left. \left. + 16\sqrt{2} \frac{f_2(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_3(\tilde{q})} - 16\sqrt{2} \frac{f_3(\tilde{q})^9 f_1(\tilde{q})}{f_4(\tilde{q})^9 f_2(\tilde{q})} \right\} \right. \\ &\quad \left. + \int_{\epsilon/4}^{\Lambda} s^{-\frac{3}{2}} ds \left\{ 2^{\frac{5}{2}} (1-i) \frac{f_3(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_4(i\tilde{q})} - 2^{\frac{5}{2}} (1+i) \frac{f_4(i\tilde{q})^9 f_1(i\tilde{q})}{f_2(i\tilde{q})^9 f_3(i\tilde{q})} \right\} \right], \end{aligned} \quad (7.2)$$

where the sum over \vec{n} refers to sum over d integers (n^1, \dots, n^d) labelling the winding numbers of open strings along the d circles. Similarly the weak coupling result (4.2) now takes the form

$$\begin{aligned} K_W &= -\frac{1}{64\pi} \int d^2\tau \int d^2z \left\{ \sum_{\nu'} \left\{ \vartheta_{\nu'} \left(\frac{z}{2} \right)^{16} \right\} (\overline{\eta(\tau)})^{-18} (\eta(\tau))^{-6} \left(\frac{\vartheta_{11}(z)}{\vartheta_{11}(z)} \right)^2 \right\} \\ &\quad \times \exp \left(-\frac{4\pi z_2^2}{\tau_2} \right) (\tau_2)^{(d-10)/2} (\det G_H)^{-1/2} \\ &\quad + \sum_{\vec{n}, \vec{w}} \exp \left[-\pi\tau_2 \left\{ (G_H^{-1})_{kl} n^k n^l + (G_H - B_H G_H^{-1} B_H)_{kl} w^k w^l + 2(G_H^{-1} B_H)_{kl} n^k w^l \right\} \right. \\ &\quad \left. - 2\pi i \tau_1 n^k w^k \right], \end{aligned} \quad (7.3)$$

where the sum over \vec{n} and \vec{w} represent the sum over d momentum quantum numbers (n^1, \dots, n^d) and d winding numbers (w^1, \dots, w^d) .

As in the case of S^1 compactification, we shall find that on the strong coupling side the computation of K_S suffers from tachyonic divergence when $2G_{Ikl} n^k n^l$ becomes less than 1

for any non-zero \vec{n} . Inside this region we need to use only the zeroth order result on the strong coupling side. On the weak coupling side, when the size of the torus T^d is small, K_W computed from (7.3) becomes large signalling an apparent breakdown of perturbation theory. The remedy is to use a T-duality transformation and use the T-dual variables. In fact here we have a large group $O(6, 6; \mathbb{Z})$ of T-duality transformation acting on the moduli space. We need to identify the analog of the regions I and II in figure 1 in which we carry out the actual computation and interpolation and then extend the result to the rest of the moduli space using heterotic T-duality invariance. Since in the compactified heterotic string theory the effective coupling constant is given by $g^2/\sqrt{\det G_H}$, the natural analog of regions I and II will be to pick that domain in the moduli space for which $\sqrt{\det G_H}$ takes the maximum possible value — i.e. given any point inside such a domain, any of its T-dual image should have lower value of $\sqrt{\det G_H}$. Once we have identified such a domain we then divide this into the two regions I and II depending on whether $2G_{I_{k\ell}}n^kn^\ell$ lies above 1 for all \vec{n} or not. The rest of the analysis would then proceed as in the case of S^1 compactification.

8 Conclusion

In this paper we have analyzed the mass formula for stable non-BPS state in type I/SO(32) heterotic string theory compactified on a circle using the interpolation formula between the strong and weak coupling results. Our analysis indicates that the interpolation formula determines the mass of the state within 10% accuracy over the entire moduli space. We also determine the region of stability of the particle based on the mass formula and discuss generalization of the analysis for generic toroidal compactification.

In recent times there has been significant developments in resumming perturbation theory [14–25]. It will be interesting to see if the interpolation between strong and weak coupling results can be combined efficiently with these resummation techniques to get a better understanding of physical quantities at intermediate values of coupling.

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References

- [1] E. Witten, *Superstring perturbation theory revisited*, [arXiv:1209.5461](https://arxiv.org/abs/1209.5461) [[INSPIRE](#)].
- [2] A. Sen, *S-duality improved superstring perturbation theory*, *JHEP* **11** (2013) 029 [[arXiv:1304.0458](https://arxiv.org/abs/1304.0458)] [[INSPIRE](#)].

- [3] C. Beem, L. Rastelli, A. Sen and B.C. van Rees, *Resummation and S-duality in $N = 4$ SYM*, [arXiv:1306.3228](#) [[INSPIRE](#)].
- [4] T. Banks and T.J. Torres, *Two point Padé approximants and duality*, [arXiv:1307.3689](#) [[INSPIRE](#)].
- [5] L.F. Alday and A. Bissi, *The superconformal bootstrap for structure constants*, [arXiv:1310.3757](#) [[INSPIRE](#)].
- [6] V. Asnin et al., *High and low dimensions in the black hole negative mode*, *Class. Quant. Grav.* **24** (2007) 5527 [[arXiv:0706.1555](#)] [[INSPIRE](#)].
- [7] H. Kleinert and V. Schulte-Frohlinde, *Critical properties of ϕ^4 -theories*, World Scientific, River Edge U.S.A. (2001).
- [8] J. Polchinski, *Dirichlet branes and Ramond-Ramond charges*, *Phys. Rev. Lett.* **75** (1995) 4724 [[hep-th/9510017](#)] [[INSPIRE](#)].
- [9] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, *Stable non-BPS D-branes in type-I string theory*, *Nucl. Phys. B* **564** (2000) 60 [[hep-th/9903123](#)] [[INSPIRE](#)].
- [10] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, *Stable non-BPS D-branes of type-I*, [hep-th/0003022](#) [[INSPIRE](#)].
- [11] M. Frau, L. Gallot, A. Lerda and P. Strigazzi, *D-branes in type-I string theory*, *Fortsch. Phys.* **49** (2001) 503 [[hep-th/0012167](#)] [[INSPIRE](#)].
- [12] A. Sen, *Non-BPS states and branes in string theory*, [hep-th/9904207](#) [[INSPIRE](#)].
- [13] J. Seo, *Exactly stable non-BPS spinors in heterotic string theory on tori*, *JHEP* **01** (2013) 051 [[arXiv:1207.5782](#)] [[INSPIRE](#)].
- [14] M. Stinger, *Field theory amplitudes as resurgent functions*, [hep-ph/0207349](#) [[INSPIRE](#)].
- [15] U.D. Jentschura and J. Zinn-Justin, *Instantons in quantum mechanics and resurgent expansions*, *Phys. Lett. B* **596** (2004) 138 [[hep-ph/0405279](#)] [[INSPIRE](#)].
- [16] I. Aniceto, R. Schiappa and M. Vonk, *The resurgence of instantons in string theory*, *Commun. Num. Theor. Phys.* **6** (2012) 339 [[arXiv:1106.5922](#)] [[INSPIRE](#)].
- [17] P.C. Argyres and M. Ünsal, *The semi-classical expansion and resurgence in gauge theories: new perturbative, instanton, bion and renormalon effects*, *JHEP* **08** (2012) 063 [[arXiv:1206.1890](#)] [[INSPIRE](#)].
- [18] G.V. Dunne and M. Ünsal, *Resurgence and trans-series in quantum field theory: the CP^{N-1} model*, *JHEP* **11** (2012) 170 [[arXiv:1210.2423](#)] [[INSPIRE](#)].
- [19] G.V. Dunne and M. Ünsal, *Continuity and resurgence: towards a continuum definition of the CP^{N-1} model*, *Phys. Rev. D* **87** (2013) 025015 [[arXiv:1210.3646](#)] [[INSPIRE](#)].
- [20] R. Schiappa and R. Vaz, *The resurgence of instantons: multi-cuts Stokes phases and the Painleve II equation*, [arXiv:1302.5138](#) [[INSPIRE](#)].
- [21] A. Cherman, D. Dorigoni, G.V. Dunne and M. Ünsal, *Resurgence in QFT: unitons, fractons and renormalons in the principal chiral model*, *Phys. Rev. Lett.* **112** (2014) 021601 [[arXiv:1308.0127](#)] [[INSPIRE](#)].
- [22] G. Basar, G.V. Dunne and M. Ünsal, *Resurgence theory, ghost-instantons and analytic continuation of path integrals*, *JHEP* **10** (2013) 041 [[arXiv:1308.1108](#)] [[INSPIRE](#)].

- [23] I. Aniceto and R. Schiappa, *Nonperturbative ambiguities and the reality of resurgent transseries*, [arXiv:1308.1115](#) [[INSPIRE](#)].
- [24] R.C. Santamaría, J.D. Edelstein, R. Schiappa and M. Vonk, *Resurgent transseries and the holomorphic anomaly*, [arXiv:1308.1695](#) [[INSPIRE](#)].
- [25] M. Garay, A. de Goursac and D. van Straten, *Resurgent deformation quantisation*, [Annals Phys.](#) **342** (2014) 83 [[arXiv:1309.0437](#)] [[INSPIRE](#)].